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Nonlinear stability of parallel flows with subcritical Reynolds numbers. Part 1. An asymptotic theory valid for small amplitude disturbances

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A strict distinction is made between the two fundamental assumptions in the Stuart–Watson theory of nonlinear stability, one of which is that the amplitude of disturbance is sufficiently small, while the other is that the damping or amplification rate for an infinitesimal disturbance is small. This distinction leads to classification of the nonlinear stability theory into two asymptotic theories: the theory based on the first assumption can be applied to subcritical flows with Reynolds numbers away from the neutral curve, even to flows with no neutral curve, such as plane Couette flow or pipe Poiseuille flow, while the theory based on the second assumption is available only for Reynolds numbers and wavenumbers in the neighbourhood of the neutral curve. In the theory based on the first assumption the concept of trajectories in phase space, together with the method of eigenfunction expansion, is introduced in order to display nonlinear behaviour of the disturbance amplitude and to provide the most rational definition of the Landau constant available for classification of the behaviour patterns.

1. Introduction

The nonlinear stability theory of two-dimensional laminar flows was founded by Stuart (1960) and Watson (1960). Following the conjecture of Landau (1944), they have shown that the behaviour of a small but finite wave disturbance is governed by the Landau-type equation

$$-\tfrac{1}{2}\,d|A|^2/dt = \mu_{1r}^{(0)}|A|^2 + \Lambda_r|A|^4 + O(|A|^6), \tag{1.1}$$

where $|A|$ is the amplitude of the disturbance, $\mu_{1r}^{(0)}$ is the damping rate for an infinitesimal disturbance and Λ_r is a quantity representing the effect of a finite disturbance and usually called the Landau constant. Numerical calculations to estimate the Landau constants for parallel or nearly parallel flows, such as plane Poiseuille flow or Blasius boundary-layer flow, have been carried out by Reynolds & Potter (1967), Pekeris & Shkoller (1967, 1969), the present author (Itoh 1974*a*, *b*) and others.

The Stuart–Watson theory is based on the following two assumptions: (1) that the amplitude $|A|$ of the disturbance is sufficiently small; (2) that the modulus of the damping rate $\mu_{1r}^{(0)}$ for an infinitesimal disturbance is also sufficiently small. Strictly speaking, the first assumption should be justified by using the asymptotic theory valid for $|A| \to 0$, and the second assumption by the asymptotic theory valid for $\mu_{1r}^{(0)} \to 0$. Accordingly, it is necessary to distinguish carefully between these two assumptions. This point of view would reveal the fact that there is a significant difference between the approach of Stuart and that of Watson.

The theory given by Stuart, and extended by Eckhaus (1965), is based on assumption 2; the damping rate $\mu_{1r}^{(0)}$ is taken as a small parameter of the expansion and the amplitude is assumed to be of order $|\mu_{1r}^{(0)}|^{\frac{1}{2}}$. This theory, however, seems to be imperfect in that coefficients of the asymptotic series are determined as functions of both the Reynolds number and the wavenumber, although the coefficients should depend upon only the Reynolds number and wavenumber on the neutral curve corresponding to the limit $\mu_{1r}^{(0)} \to 0$, as suggested by Stuart himself. The rigorous formulation is given in Stewartson & Stuart (1971), where all quantities are expanded about the minimum critical point on the neutral curve. Since assumption 2 requires the existence of a linear neutral curve, this approach is available only for such problems as plane Poiseuille flow and Blasius flow, and cannot be applied to subcritical flows with no neutral curve, such as plane Couette flow or pipe Poiseuille flow.

On the other hand, the approach of Watson is *by its nature* based on assumption 1. However, he assumed that each Fourier component of the stream function for the disturbance with respect to the flow direction is determined uniquely as a function of the fundamental component. This treatment is permissible under assumption 2 and therefore turns out to restrict the validity of his approach to a narrow range of the Reynolds number R and wavenumber α in the vicinity of the neutral curve. Actually, Ellingsen, Gjevik & Palm (1970, § 4) applied this approach to plane Couette flow, for which $\mu_{1r}^{(0)}$ is always positive in the whole range of R and α, and obtained the peculiar result that the sign of the Landau constant changed quite rapidly with variation of R and α. This fact suggests that the approach of Watson is not applicable to ranges of R and α away from the neutral curve. On the other hand, another method of approximation adopted by Ellingsen *et al.* (1970, § 5) leads to the apparently plausible result that the Landau constant retains a fixed sign for a large range of R and α. The essential point of this approximation is to neglect the damping rates of both the mean-flow distortion and the second-harmonic component, but no convincing justification has been provided for this approximation. It is the aim of the present paper to indicate that the asymptotic theory based on assumption 1 gives rise to the same equations defining the Landau constant as those given by Ellingsen *et al.* and that the Landau constant defined in this way is meaningful for any Reynolds number and wavenumber and is very powerful for classifying the behaviour patterns of disturbances. This aim is achieved by using the method of eigenfunction expansion developed by Eckhaus (1965) and the technique of trajectories in phase space.

2. Expansion procedure

We restrict our attention to two-dimensional problems. Let (x, y) be the non-dimensional co-ordinates in the streamwise and normal directions, respectively, and t be the time. The stream function $\Psi(x, y, t)$ of a disturbance superimposed on a basic laminar flow with velocity $U(y)$ is governed by the equation

$$(L - \partial M/\partial t)\Psi = \tfrac{1}{2}N^*[\Psi, \Psi], \tag{2.1}$$

where the operators L, M and N^* are defined by

$$L \equiv \frac{1}{R}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)^2 - \left\{U(y)\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right) - \frac{d^2U(y)}{dy^2}\right\}\frac{\partial}{\partial x}, \quad M \equiv \frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2},$$

$$N^*[\Psi, \Phi] \equiv \left(\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)\Phi + \left(\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial x^2}\right)\Psi,$$

and R is the Reynolds number of the flow. We consider a periodic solution of the above equation, i.e. a solution representing a monochromatic wave with wavenumber α in the x direction. Thus Ψ may be written as a Fourier series:

$$\Psi(x,y,t) = \sum_{k=-\infty}^{\infty} \Psi_k(y,t)\, e^{ik\alpha x}, \tag{2.2}$$

where $\Psi_{-k} = \tilde{\Psi}_k$, the tilde denoting the complex conjugate. Substituting (2.2) into (2.1) and separating out the Fourier components, we have an infinite set of equations:

$$\left(L_k - \frac{\partial}{\partial t}M_k\right)\Psi_k = \frac{1}{2}\sum_{l=-\infty}^{\infty} N[\Psi_{k-l}, \Psi_l], \tag{2.3}$$

where $k = 0, \pm 1, \pm 2, \ldots$, and the operators L_k and M_k are those obtained by replacing $\partial/\partial x$ with $ik\alpha$ in the operators L and M while N is that obtained by replacing $\partial/\partial x$ with $i(k-l)\,\alpha$ or $il\alpha$ in the operator N^*.

In the linear problem, where the right-hand side of (2.3) is ignored, the solution is assumed to be of the form

$$\Psi_k(y,t) = \phi_k(y) \exp(-\mu_k t),$$

which leads to the Orr–Sommerfeld equation

$$(L_k + \mu_k M_k)\,\phi_k(y) = 0. \tag{2.4}$$

This equation, together with homogeneous boundary conditions, provides an eigenvalue problem determining μ_k as a function of R and $k\alpha$. The real and imaginary parts of the eigenvalue represent the damping rate and frequency of the corresponding eigenmode in the disturbance. For each value of k there exist an infinite sequence of eigenvalues $\mu_k^{(n)}$ $(n = 0, 1, 2, \ldots)$, ordered in such a way that $\mathrm{Re}\,\mu_k^{(n+1)} > \mathrm{Re}\,\mu_k^{(n)}$, and the corresponding eigenfunctions $\phi_k^{(n)}(y)$, which are normalized in a suitable manner and are assumed to constitute a complete system. If we introduce the adjoint eigenfunctions $\hat{\phi}_k^{(m)}(y)$ $(m = 0, 1, 2, \ldots)$ in a normalized form, then an orthogonality relation can be obtained (Eckhaus 1965, chap. 6):

$$\int_0^1 \hat{\phi}_k^{(m)} M_k \phi_k^{(n)}\, dy = \delta_{mn}, \tag{2.5}$$

where δ_{mn} denotes the Kronecker delta.

Using the above results of the linear theory and regarding the right-hand side of (2.3) as a forcing term, we may write the formal solution of (2.3) in the form

$$\Psi_k(y,t) = \sum_{n=0}^{\infty} A_k^{(n)}(t)\, \phi_k^{(n)}(y). \tag{2.6}$$

If we substitute (2.6) into (2.3), multiply by the adjoint eigenfunction $\hat{\phi}_k^{(m)}(y)$ and integrate the resultant equation with respect to y from 0 to 1, an infinite set of simultaneous equations for the unknown amplitude functions $A_k^{(m)}(t)$ is obtained:

$$\frac{dA_k^{(m)}}{dt}+\mu_k^{(m)}A_k^{(m)} = -\frac{1}{2}\sum_{l=-\infty}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sigma_{kl}^{(m,p,q)}A_{k-l}^{(p)}A_l^{(q)}, \tag{2.7}$$

where $A_{-k}^{(m)} = \tilde{A}_k^{(m)}$ and

$$\sigma_{kl}^{(m,p,q)} = \int_0^1 \hat{\phi}_k^{(m)}N[\phi_{k-l}^{(p)},\phi_l^{(q)}]\,dy. \tag{2.8}$$

The next step is to solve the initial-value problem for the temporal variation of the amplitude governed by the above equations. Here it should be noted that what is interesting to us is not a particular solution under some specific initial conditions, but to obtain the most realistic solution when the disturbance is sufficiently small. If appropriate orders of magnitude of the amplitude functions are defined by initial conditions and all the amplitude functions determined from the solution of (2.7) are found to remain of the same orders as the initial conditions during the time history, then the initial conditions and also the solution are considered to be natural. For such a treatment the order-of-magnitude considerations introduced by Stuart (1960) and Watson (1960) are available. If we assume that the first eigenvalue $\mu_1^{(0)}$ of the fundamental component has the smallest damping rate among all eigenvalues

$$\mu_k^{(n)} \quad (k,n = 0,1,2,\ldots),$$

the following estimation of the orders of magnitude leads to no inconsistency:

$$\left.\begin{aligned}&A_1^{(0)}(t) = \epsilon a_1^{(0)}(t),\quad A_0^{(n)}(t) = \epsilon^2 a_0^{(n)}(t),\quad A_2^{(n)}(t) = \epsilon^2 a_2^{(n)}(t)\quad (n = 0,1,2,\ldots),\\ &A_1^{(n)}(t) = O(\epsilon^3)\quad (n\geqslant 1),\quad A_k^{(n)}(t) = O(\epsilon^k)\quad (k\geqslant 3,\quad n\geqslant 0),\end{aligned}\right\} \tag{2.9}$$

where $a_k^{(n)}(t)$ is of order unity and ϵ is a small quantity representing the magnitude of the first eigenmode $A_1^{(0)}(t)$ of the fundamental component, the other eigenmodes $A_1^{(n)}(t)$ being taken to be of order ϵ^3 because they result from nonlinear interactions between $A_1^{(0)}$ and $A_0^{(n)}$ or $A_2^{(n)}$. Here it is convenient to decompose the $a_k^{(n)}$ $(k \neq 0)$ into the real amplitudes and phase angles, because the $a_k^{(n)}$ $(k\neq 0)$ are complex while $a_0^{(n)}$ is real. Moreover, we need only the phase angles relative to the fundamental component, and we introduce the phase differences defined by

$$\Theta_k^{(n)}(t) = \theta_k^{(n)}(t)-k\theta_1^{(0)}(t), \tag{2.10}$$

where

$$a_k^{(n)}(t) = |a_k^{(n)}(t)|\exp\{i\theta_k^{(n)}(t)\} \tag{2.11}$$

and $k = 1, 2, 3, \ldots, n = 0, 1, 2, \ldots$. Substituting (2.9)–(2.11) into (2.7) and separating out the real and imaginary parts, we have

$$\begin{aligned}\frac{d|a_1^{(0)}|^2}{dt} &= -2\mu_{1r}^{(0)}|a_1^{(0)}|^2-2\epsilon^2|a_1^{(0)}|^2\\ &\quad\times\sum_{p=0}^{\infty}\{\sigma_{10r}^{(0,0,p)}a_0^{(p)}+|\sigma_{12}^{(0,0,p)}|\,|a_2^{(p)}|\cos(\Theta_2^{(p)}+\rho_{12}^{(0,0,p)})\}+O(\epsilon^4),\end{aligned} \tag{2.12}$$

$$\frac{da_0^{(n)}}{dt} = -\mu_0^{(n)}a_0^{(n)}-\sigma_{01}^{(n,0,0)}|a_1^{(0)}|^2+O(\epsilon^2), \tag{2.13}$$

$$\frac{d|a_2^{(n)}|}{dt} = -\mu_{2r}^{(n)}|a_2^{(n)}|-\tfrac{1}{2}|a_1^{(0)}|^2|\sigma_{21}^{(n,0,0)}|\cos(\Theta_2^{(n)}-\rho_{21}^{(n,0,0)})+O(\epsilon^2), \tag{2.14a}$$

$$\frac{d\Theta_2^{(n)}}{dt} = -(\mu_{2i}^{(n)}-2\mu_{1i}^{(0)})+\frac{1}{2}\frac{|a_1^{(0)}|^2}{|a_2^{(n)}|}|\sigma_{21}^{(n,0,0)}|\sin(\Theta_2^{(n)}-\rho_{21}^{(n,0,0)})+O(\epsilon^2), \tag{2.14b}$$

where $\sigma_{kl}^{(n,p,q)} = |\sigma_{kl}^{(n,p,q)}| \exp(i\rho_{kl}^{(n,p,q)})$, suffixes r and i denote the real and imaginary parts, respectively, while $\mu_0^{(n)}$ and $\sigma_{01}^{(n,0,0)}$ are real. We have ignored the equations for the other components, which are of smaller order. The above equations define a group of trajectories in an infinite-dimensional phase space with co-ordinates $|a_1^{(0)}|^2$, $a_0^{(n)}$, $|a_2^{(n)}|$ and $\Theta_2^{(n)}$ $(n = 0, 1, 2, \ldots)$, the time t being considered as a parameter. In the next section we use a topological consideration to classify the shape of trajectories in the phase space.

3. Trajectories in the phase space

By equating the right-hand sides of (2.12)–(2.14) to zero, we obtain surfaces in the infinite-dimensional phase space with co-ordinates

$$|a_1^{(0)}|^2, \quad a_0^{(n)}, \quad |a_2^{(n)}| \quad \text{and} \quad \Theta_2^{(n)} \quad (n = 0, 1, 2, \ldots).$$

On each surface one of the amplitude functions is in equilibrium in the sense that its derivative with respect to time is equal to zero. In other words, when a trajectory crosses one of the surfaces, its tangent has the direction normal to the co-ordinate for which the surface is defined. Therefore, if all the surfaces intersect at a point, it is an equilibrium point of the phase space. In order to reveal the properties of trajectories it is necessary to investigate the shape of the surfaces and the position and stability of the equilibrium points.

First we examine properties in the limiting case $\epsilon \to 0$ of the surfaces

$$a_0^{(n)} = -\frac{\sigma_{01}^{(n,0,0)}}{\mu_0^{(n)}} |a_1^{(0)}|^2 + O(\epsilon^2), \tag{3.1}$$

$$|a_2^{(n)}| = \frac{|\sigma_{21}^{(n,0,0)}|}{[(\mu_{2r}^{(n)})^2 + (\mu_{2i}^{(n)} - 2\mu_{1i}^{(0)})^2]^{\frac{1}{2}}} \frac{|a_1^{(0)}|^2}{2} + O(\epsilon^2), \tag{3.2a}$$

$$\Theta_2^{(n)} = \rho_{21}^{(n,0,0)} + \tan^{-1}\left(-\frac{\mu_{2i}^{(n)} - 2\mu_{1i}^{(0)}}{\mu_{2r}^{(n)}}\right) + O(\epsilon^2), \tag{3.2b}$$

which are obtained by equating the right-hand sides of (2.13) and (2.14) to zero. In the limiting case $\epsilon \to 0$ each of equations (3.1) represents a plane surface through the origin of the space, and the tangent to every trajectory at its intersection with this plane is found to be parallel to the co-ordinate axis $|a_1^{(0)}|^2$. Similarly, equations (3.2a) define plane surfaces (with constant values (3.2b) of $\Theta_2^{(n)}$) on which trajectories have tangents parallel to the $|a_1^{(0)}|^2$ axis. Therefore we can easily see that all surfaces (3.1) and (3.2) intersect along a straight line on which $a_0^{(n)}$ and $|a_2^{(n)}|$ are proportional to $|a_1^{(0)}|^2$ and $\Theta_2^{(n)}$ is constant. If there are equilibrium points in this phase space, they must all be located on this straight line.

On the other hand, if we put $\epsilon = 0$ in (2.12), the surface on which $d|a_1^{(0)}|^2/dt = 0$ is given by $|a_1^{(0)}|^2 = 0$, so long as $\mu_{1r}^{(0)} \neq 0$. [The case when $\mu_{1r}^{(0)}$ is very small is discussed in § 5.] The intersection of this surface with the straight line defined by (3.1) and (3.2) is at the origin of the phase space $|a_1^{(0)}|^2 = a_0^{(n)} = |a_2^{(n)}| = 0$, and the stability condition for this equilibrium point is that $\mu_{1r}^{(0)} > 0$, as predicted by linear theory.

Next we consider the case when ϵ is small but finite. Then the surfaces (3.1) and (3.2) are not planes, but somewhat curved surfaces slightly distorted from plane surfaces. Accordingly the intersection line likewise becomes a curve slightly distorted from a straight line. The damping rate of the fundamental wave along this curve is obtained from substitution of (3.1) and (3.2) into (2.12) as

$$d|a_1^{(0)}|^2/dt = -2|a_1^{(0)}|^2\{\mu_{1r}^{(0)} + \epsilon^2\lambda_r|a_1^{(0)}|^2 + O(\epsilon^4)\}, \tag{3.3}$$

where λ_r is the real part of the complex quantity λ defined by

$$\lambda = \sum_{p=0}^{\infty} \frac{\sigma_{10}^{(0,0,p)}\sigma_{01}^{(p,0,0)}}{-\mu_0^{(p)}} + \frac{1}{2}\sum_{p=0}^{\infty} \frac{\sigma_{12}^{(0,0,p)}\sigma_{21}^{(p,0,0)}}{2i\mu_{1i}^{(0)} - \mu_2^{(p)}}. \tag{3.4}$$

The equilibrium points are given by the values of $|a_1^{(0)}|^2$ which make the right-hand side of (3.3) vanish. One of the possibilities is given by the case $|a_1^{(0)}|^2 = 0$, which has already been discussed, and another possibility is formally given by

$$|a_1^{(0)}|^2 = -\mu_{1r}^{(0)}/\epsilon^2\lambda_r + O(\epsilon^2). \tag{3.5}$$

This point, however, will move out to infinity in the limit $\epsilon \to 0$, so long as $\mu_{1r}^{(0)} \neq 0$ and $\lambda_r \neq \infty$, $\mu_{1r}^{(0)}/\lambda_r$ being finite, i.e. independent of ϵ. This contradicts the assumption that $a_1^{(0)}(t)$ is of order unity. In other words, the asymptotic theory with the disturbance amplitude as the small parameter turns out not to provide an estimate of finite equilibrium amplitudes. Nevertheless, it is certain that this approach discloses some features of trajectories in the phase space for sufficiently small values of ϵ. Even if ϵ tends to zero, so that the equilibrium point given by (3.5) moves out to infinity, we can see the shape of trajectories at least in the region where $|a_1^{(0)}|^2$, $a_0^{(n)}$ and $|a_2^{(n)}|$ ($n = 0, 1, 2, \ldots$) are all of order unity. Moreover, it is found that the shape of trajectories can be classified into four patterns according to the four combinations of the signs of $\mu_{1r}^{(0)}$ and λ_r.

In order to comprehend the features of trajectories, we restrict our attention for the moment to the simpler case of a two-dimensional phase plane with co-ordinates $|a_{1r}^{(0)}|^2$ and $a_0^{(0)}$, and investigate the shape of trajectories governed by

$$d|a_1^{(0)}|^2/dt = -2|a_1^{(0)}|^2(\mu_{1r}^{(0)} + \epsilon^2\sigma_{10r}^{(0,0,0)}a_0^{(0)}), \tag{3.6}$$

$$da_0^{(0)}/dt = -\mu_0^{(0)}a_0^{(0)} - \sigma_{01}^{(0,0,0)}|a_1^{(0)}|^2, \tag{3.7}$$

where smaller terms $O(\epsilon^4)$ in (3.6) and $O(\epsilon^2)$ in (3.7) have been ignored, $\mu_0^{(0)}$ is assumed to be positive on the basis of the fact that linear solutions for the mean-flow distortion decay in most problems of stability, while other coefficients $\mu_{1r}^{(0)}$, $\sigma_{10r}^{(0,0,0)}$ and $\sigma_{01}^{(0,0,0)}$ may be positive or negative. In this case the definition (3.4) of λ_r reduces to

$$\lambda_r = \sigma_{10r}^{(0,0,0)}\sigma_{01}^{(0,0,0)}/(-\mu_0^{(0)}).$$

Figure 1 shows four possible patterns of trajectories in the $|a_1^{(0)}|^2$, $a_0^{(0)}$ plane, which are classified according to the signs of the linear damping rate $\mu_{1r}^{(0)}$ and the coefficient λ_r. Here the straight lines PC and OD denote solutions of the equations obtained by equating the right-hand sides of (3.6) and (3.7) to zero, respectively, C being the intersection of the two lines. In the case when $\mu_{1r}^{(0)} > 0$ and $\lambda_r < 0$, the straight line OD on which $da_0^{(0)}/dt = 0$ intersects the straight line PC on which $d|a_1^{(0)}|^2/dt = 0$ at a point located in the region $|a_1^{(0)}|^2 > 0$. Noting that $\mu_0^{(0)}$ is positive, so that $da_0^{(0)}/dt$ is positive on the lower side and negative on the upper side of OD, and that $\mu_{1r}^{(0)}$ is positive, so that $d|a_1^{(0)}|^2/dt$ is negative on the same side as the origin O and positive on the opposite side of PC, we can draw the rough shape of trajectories, as shown in figure 1(a). In

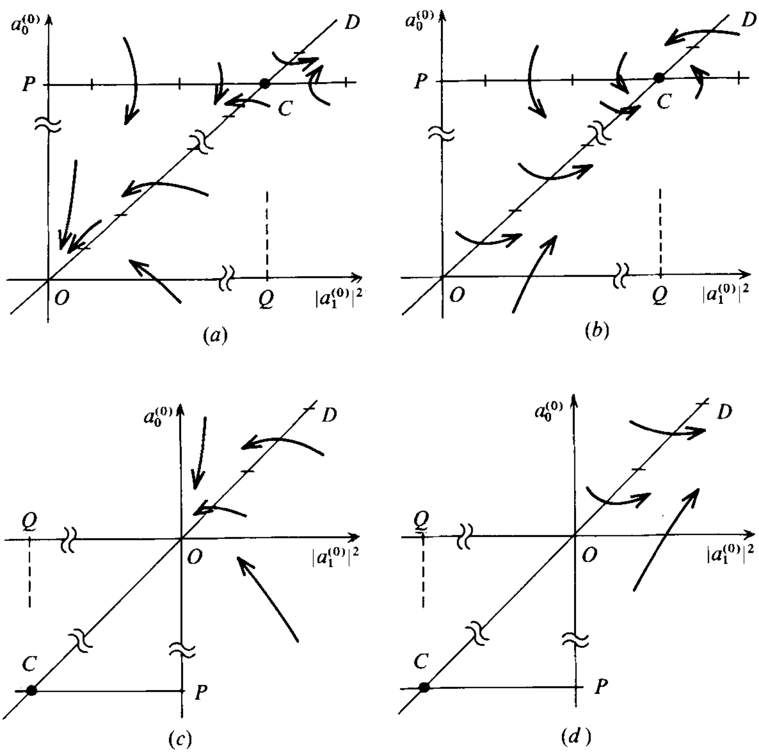

FIGURE 1. Four patterns of trajectories on the phase plane. (*a*) $\mu_{1r}^{(0)} > 0$, $\lambda_r < 0$. (*b*) $\mu_{1r}^{(0)} < 0$, $\lambda_r > 0$. (*c*) $\mu_{1r}^{(0)} > 0, \lambda_r > 0$. (*d*) $\mu_{1r}^{(0)} < 0$, $\lambda_r < 0$. ($P = -\mu_{1r}^{(0)}/(\epsilon^2\sigma_{10r}^{(0,0,0)})$, $Q = -\mu_{1r}^{(0)}/(\epsilon^2\lambda_r)$.)

this case the equilibrium point C is found to be unstable, having a large positive value of $|a_1^{(0)}|^2$ of order $1/\epsilon^2$. This shape of trajectories shows that a disturbance decays when it has a sufficiently small initial amplitude, but may grow when it has a sufficiently large initial amplitude, say $O(1/\epsilon^2)$. On the other hand, the case when $\mu_{1r}^{(0)} < 0$ and $\lambda_r > 0$ shows the existence of a stable equilibrium point with a large positive value of $|a_1^{(0)}|^2$, and trajectories tend to this point as the parameter t increases. In the other two cases the equilibrium points are located in the physically meaningless region $|a_1^{(0)}|^2 < 0$, and trajectories in the region $|a_1^{(0)}|^2 \geqslant 0$ show that disturbances monotonically decay or grow according as $\mu_{1r}^{(0)}$ and λ_r are both positive or negative.

The above classification of trajectories into four patterns can be applied directly to the case of the infinite-dimensional phase space defined by (2.12)–(2.14), because of the orthogonality of all the co-ordinate axes. It is however necessary to note that (3.2*b*) defines two values of $\Theta_2^{(n)}$ in the range $-\pi < \Theta_2^{(n)} < \pi$. We should choose the value which makes the corresponding surface represent a stable equilibrium to small shifts in $|a_2^{(n)}|$ and $\Theta_2^{(n)}$ from the values given by (3.2*a*, *b*). Since the quantities necessary for the classification are $\mu_{1r}^{(0)}$ and λ_r only, it can be considered that the classification is summarized by (3.3). A noteworthy fact is that (3.3) is an asymptotic expression (valid for small amplitude) for the damping rate of the fundamental wave on *a curve* representing the solution of the simultaneous equations (3.1) and (3.2). When the damping rate of a disturbance is expressed as a power series in the amplitude, the second coefficient is usually called the Landau constant and represents the dominant effect of nonlinearity. In order to rewrite (2.12) in the form of (3.3), however, we need to determine only a straight line passing through the origin in the phase space. Therefore there are an infinite number of definitions of the Landau constant corresponding to the infinite number of choices of the straight line. However, the above investigation shows that the definition (3.4) is most useful from the viewpoint that the Landau constant should be a basic factor for classification of trajectories in the phase space. This point is discussed further in § 4.

Here a simpler method is shown to yield the complex Landau constant λ defined by (3.4) without use of eigenfunction expansions. By applying (2.8) to $\sigma_{10}^{(0,0,p)}$ and $\sigma_{12}^{(0,0,p)}$, we may rewrite the definition (3.4) in the form

$$\lambda = \int_0^1 \tilde{\phi}_1^{(0)}(y)\, N[\phi_1^{(0)}(y), g_0(y)]\, dy + \int_0^1 \tilde{\phi}_1^{(0)}(y)\, N[\bar{\phi}_1^{(0)}(y), g_2(y)]\, dy, \tag{3.8}$$

where
$$g_0(y) = \sum_{p=0}^{\infty} \frac{\sigma_{01}^{(p,0,0)}}{-\mu_0^{(p)}} \phi_0^{(p)}(y), \quad g_2(y) = \frac{1}{2} \sum_{p=0}^{\infty} \frac{\sigma_{21}^{(p,0,0)}}{2i\mu_{1i}^{(0)} - \mu_2^{(p)}} \phi_2^{(p)}(y). \tag{3.9}$$

Then it is easily found from the definition of $\sigma_{kl}^{(m,p,q)}$ that the functions $g_0(y)$ and $g_2(y)$ are the solutions of the equations

$$L_0 g_0(y) = N[\phi_1^{(0)}(y), \bar{\phi}_1^{(0)}(y)] \tag{3.10}$$

and
$$(L_2 + 2i\mu_{1i}^{(0)} M_2) g_2 = \tfrac{1}{2} N[\phi_1^{(0)}(y), \phi_1^{(0)}(y)], \tag{3.11}$$

respectively. Thus we can obtain the value of λ by solving (3.10) and (3.11), substituting the solutions into the integrands of (3.8), and performing the integrations. Equations (3.10) and (3.11) correspond to equations (5.1) and (5.2) in the paper of Ellingsen *et al.* (1970), respectively.

4. Comparison with the method of Watson

The equation (2.13) for the mean-flow distortion has the solution

$$a_0^{(n)}(t) = C_0^{(n)} \exp(-\mu_0^{(n)} t) - \sigma_{01}^{(n,0,0)} \exp(-\mu_0^{(n)} t) \int^t |a_1^{(0)}(t')|^2 \exp(\mu_0^{(n)} t')\, dt' + O(\epsilon^2), \tag{4.1}$$

where $C_0^{(n)}$ is an arbitrary constant. In the formulation of Watson (1960), the first term, containing $C_0^{(n)}$, is ignored because every damping rate $\mu_0^{(n)}$ for the corresponding eigenmode of the mean-flow distortion is much larger than the damping rate of the fundamental wave. [A clearer explanation of this process is given by Eckhaus (1965, chap. 7).] If we follow his approach with the same treatment for the solution of the second-harmonic component, we obtain the solutions of (2.13) and (2.14) as

$$a_0^{(n)}(t) = \frac{\sigma_{01}^{(n,0,0)}}{2\mu_{1r}^{(0)} - \mu_0^{(n)}} |a_1^{(0)}(t)|^2 + O(\epsilon^2), \tag{4.2}$$

$$a_2^{(n)}(t) = \frac{1}{2} \frac{\sigma_{21}^{(n,0,0)}}{2\mu_1^{(0)} - \mu_2^{(n)}} \{a_1^{(0)}(t)\}^2 + O(\epsilon^2). \tag{4.3}$$

Substitution of these solutions, instead of (3.1) and (3.2), into (2.12) leads to another definition of the Landau constant:

$$\Lambda = \int_0^1 \tilde{\phi}_1^{(0)}(y)\, N[\phi_1^{(0)}(y), G_0(y)]\, dy + \int_0^1 \tilde{\phi}_1^{(0)}(y)\, N[\bar{\phi}_1^{(0)}(y), G_2(y)]\, dy, \tag{4.4}$$

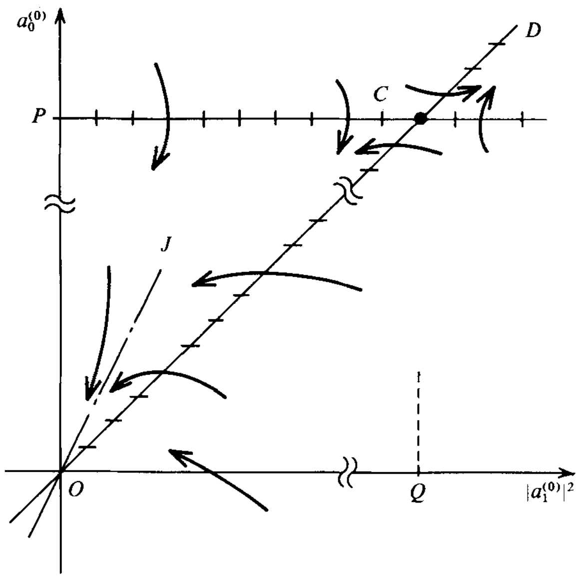

FIGURE 2. Trajectories in the case $\mu_{1r}^{(0)} > 0$ and $\lambda_r < 0$.

where the functions $G_0(y)$ and $G_2(y)$ are the solutions of the equations

$$(L_0 + 2\mu_{1r}^{(0)} M_0)\, G_0(y) = N[\phi_1^{(0)}(y), \tilde{\phi}_1^{(0)}(y)] \tag{4.5}$$

and

$$(L_2 + 2\mu_1^{(0)} M_2)\, G_2(y) = \tfrac{1}{2}N[\phi_1^{(0)}(y), \phi_1^{(0)}(y)], \tag{4.6}$$

respectively. Equations (4.5) and (4.6) differ from (3.10) and (3.11) only by the additional terms $2\mu_{1r}^{(0)} M_0$ and $2\mu_{1r}^{(0)} M_2$ involved in the operators on the left-hand sides.

Thus we have two different definitions of the Landau constant, one of which is given by (3.8) with (3.10) and (3.11) and the other by (4.4) with (4.5) and (4.6). Here let us compare these two Landau constants λ and Λ by the use of phase-space considerations. Again we take the simpler example of the two-dimensional phase plane, as shown in figure 2. Then the quantity λ is, as we have seen, defined in association with the straight line $da_0^{(0)}/dt = 0$ (namely OD in figure 2), while Λ is defined in association with the asymptote OJ on which the trajectory approaches the origin, as can be seen by obtaining the direction from (3.6) and (3.7) and comparing with (4.2). Although the line OJ coincides with OD in the neutral case $\mu_{1r}^{(0)} = 0$, the separation of the two lines owing to a non-zero value of $\mu_{1r}^{(0)}$ leads to the difference between λ and Λ. In particular, when $2\mu_{1r}^{(0)}$ or $2\mu_1^{(0)}$ is equal to one of the eigenvalues $\mu_0^{(n)}$ or $\mu_2^{(n)}$, the Landau constant Λ becomes infinite because of singularities in (4.2) and (4.3) (cf. Ellingsen *et al.* 1970). This is considered as the main reason why the method of Watson is invalid for subcritical flows, as pointed out by Davey & Nguyen (1971). If an equilibrium point with finite amplitude exists in the phase plane, it must be located on the line OD, and not on the asymptote OJ. Since the position of equilibrium points governs the main behaviour of trajectories, the Landau constant λ defined in association with the straight line OD is the most rational quantity for representing the most important properties of nonlinear development of disturbances.

For a clearer illustration of the above description, it is relevant to take an example

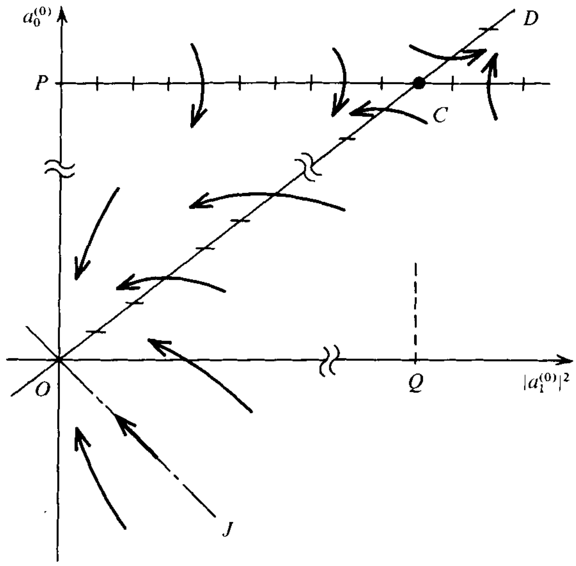

FIGURE 3. Trajectories in the case $0 < \mu_{1r}^{(0)} < \mu_0^{(0)} < 2\mu_{1r}^{(0)} \ll 1$.

in which Λ_r has the opposite sign to λ_r. In the two-dimensional problem with co-ordinates $|a_1^{(0)}|^2$ and $a_0^{(0)}$, we consider the case where $\mu_0^{(0)}$ and $\mu_{1r}^{(0)}$ satisfy the relation

$$0 < \mu_{1r}^{(0)} < \mu_0^{(0)} < 2\mu_{1r}^{(0)} \ll 1. \tag{4.7}$$

[We can easily find some examples satisfying this relation. For instance, for plane Poiseuille flow at $R = 5000$ with $\alpha = 0{\cdot}95$, the author's recent calculations show that $\mu_{1r}^{(0)} = 0{\cdot}00284$ and $\mu_0^{(0)} = 0{\cdot}00404$.] Supposing that both $\sigma_{01}^{(0,0,0)}$ and $\sigma_{10r}^{(0,0,0)}$ are negative, we obtain the trajectories on the phase plane shown in figure 3. The gradient of the straight line OD, given by $-\sigma_{01}^{(0,0,0)}/\mu_0^{(0)}$, is positive, while that of OJ, given by

$$\sigma_{01}^{(0,0,0)}/(2\mu_{1r}^{(0)} - \mu_0^{(0)}),$$

is negative. Thus the two Landau constants are opposite in sign:

$$\lambda_r = \frac{\sigma_{10r}^{(0,0,0)}\,\sigma_{01}^{(0,0,0)}}{-\mu_0^{(0)}} < 0, \quad \Lambda_r = \frac{\sigma_{10r}^{(0,0,0)}\,\sigma_{01}^{(0,0,0)}}{2\mu_{1r}^{(0)} - \mu_0^{(0)}} > 0. \tag{4.8}$$

The Landau constant Λ_r defined by (4.4)–(4.6) apparently suggests a damping effect of finite amplitude disturbances, but the trajectories in figure 3 belong to the first pattern in figure 1, showing that the effect of finite amplitude is to reduce the damping of disturbances. This example clearly indicates that λ_r can precisely describe the variation in disturbance amplitude, while Λ_r is inadequate for this purpose. We note in this example that the line OJ is not even an asymptote because of the condition $\mu_0^{(0)} < 2\mu_{1r}^{(0)}$, although it is one of the trajectories.

Here a remark should be made on another important difference between the two approaches mentioned above. When the Landau constant defined by Watson is used, an amplitude equation of the form (1.1) governs the variation in disturbance amplitude along a trajectory in the neighbourhood of the origin. On the other hand, equation (3.3), where λ_r is used, exhibits only the damping rate of the fundamental wave on the

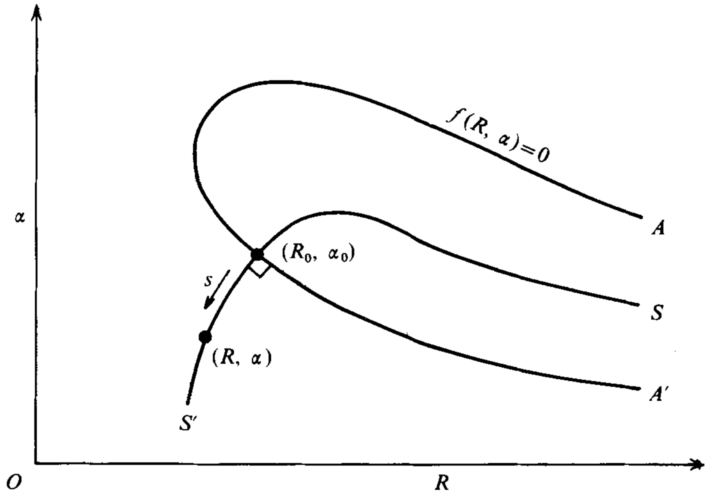

FIGURE 4. The neutral curve $\mu_{1r}^{(0)} \equiv f(R, \alpha) = 0$ and an example of the path along which the distance s is measured.

line OD defined by (3.1) and (3.2), and does not provide any information on the variation in disturbance amplitude along trajectories. The Landau constant λ_r is meaningful in the sense that it is a *key* to classification of the four patterns in the phase space.

5. Comparison with the theory of Stuart

As mentioned in § 3, the asymptotic expansion with respect to a small amplitude does not contribute to determination of finite equilibrium amplitudes. However, for a very small value of $\mu_{1r}^{(0)}$, say $O(\epsilon^2)$, (3.5) seems to be a good approximation to a true equilibrium amplitude. Stuart (1960) has chosen the damping rate $\mu_{1r}^{(0)}$ as a small expansion parameter together with the assumption that the amplitude $|A_1^{(0)}|$ is of order $|\mu_{1r}^{(0)}|^{\frac{1}{2}}$ (i.e. $\epsilon^2 = O(\mu_{1r}^{(0)})$). This approach corresponds to an asymptotic expansion with respect to the small distance s from a point (R_0, α_0) on the neutral curve along an appropriate path on the R, α plane, as shown schematically in figure 4. Although the path may be chosen arbitrarily, the most convenient one seems to be the curve crossing every iso-damping line normally. Once the path has been determined, the quantities R, α and $\mu_1^{(0)}$ can be expanded in power series in the distance s as

$$R = R_0 + (dR/ds)_0\, s + O(s^2), \tag{5.1}$$

$$\alpha = \alpha_0 + (d\alpha/ds)_0\, s + O(s^2), \tag{5.2}$$

$$\mu_1^{(0)} = (\mu_1^{(0)})_0 + (d\mu_1^{(0)}/ds)_0\, s + O(s^2), \tag{5.3}$$

where a subscript zero denotes the value at the point (R_0, α_0). Here $(d\mu_1^{(0)}/ds)_0$ is obtained from the solvability condition for the equation derived from substituting (5.1) and (5.2) into (2.4) with $k = 1$ and expanding the solution in powers of s (see Stewartson & Stuart 1971). Since the real part of $(\mu_1^{(0)})_0$ is zero, we see that $\mu_{1r}^{(0)}$ is of order s. Moreover, on the basis of the assumption that the magnitude ϵ of the disturbance is of order $s^{\frac{1}{2}}$, we may put

$$\epsilon^2 = s, \tag{5.4}$$

because there are no other restrictions on the selection of ϵ. Substitution of (5.1)–(5.4) into all the equations of §§ 2 and 3 leads to a complete asymptotic theory valid for a small value of s, the Landau constant also being written in the form of a power series:

$$\lambda_r = (\lambda_r)_0 + O(s). \tag{5.5}$$

This theory provides the asymptotic expression for a finite equilibrium amplitude as

$$|a_1^{(0)}|^2 = -\frac{(d\mu_{1r}^{(0)}/ds)_0}{(\lambda_r)_0} + O(s), \tag{5.6}$$

which tends to a finite value as $s \to 0$. The patterns of trajectories in the phase space are the same as those in figure 1 except that the equilibrium point remains in a finite range of $|a_1^{(0)}|^2$ even if $s \to 0$, although we should replace $\mu_{1r}^{(0)}$ and λ_r with $(d\mu_{1r}^{(0)}/ds)_0 s$ and $(\lambda_r)_0$, respectively.

The above asymptotic theory, which forms a rigorous extension of the theory of Stuart, should be precisely distinguished from the theory given in §§ 2 and 3. In the former the coefficients in the expansion depend upon only a point on the neutral curve, the distance from which is taken as the expansion parameter, so that the theory is valid only in the vicinity of the neutral curve. A great advantage of expansion with respect to the distance from the neutral curve is that it presents an asymptotic expression for a finite equilibrium amplitude, which cannot be derived from the expansion with respect to the amplitude of disturbance. On the other hand, the theory presented in §§ 2 and 3 is valid for any Reynolds number and wavenumber, and is therefore applicable to subcritical flows with no neutral curve, such as plane Couette flow or pipe Poiseuille flow, because the coefficients in the expansion are determined for a given Reynolds number, wavenumber pair, which is not required to be located in the vicinity of the neutral curve. However, the wavenumber should be chosen so as to produce the smallest or nearly smallest damping for the given Reynolds number. This selection would provide a basis for the assumption that natural disturbances can be suitably described by the order-of-magnitude considerations of (2.9). It is also possible to examine interactions of two or more harmonic components by the use of similar phase-space techniques if we introduce an appropriate ordering different from (2.9), for instance $A_1^{(0)}(t) = O(\epsilon)$ and $A_2^{(0)}(t) = O(\epsilon)$, as assumed by Craik (1971) for an investigation of resonant triads of Tollmien–Schlichting waves.

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